Appendix to *Efficient Timing of Retirement:*

An Approximate Solution to the Pre-Retirement Problem

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1. INTRODUCTION

Post retirement, the model in the main text[1] reduces to the Merton (1969) problem, which has of course an exact solution. Pre-retirement, however, the agent holds an American option, namely, retire now or keep working. Problems involving American options are generally difficult to solve exactly. This appendix describes an approximate solution to the agent's pre-retirement problem.

2. VALUE FUNCTIONS

Following e.g. Stock and Wise (1990) and Sundaresan and Zapatero (1997) the basis of the approximation procedure used in this appendix is the notion of *retirement precommitment*, as distinct from the *retirement flexibility* assumed in the main text. (This is closely related to the distinction between labor supply flexibility and labor supply precommitment due to Bodie, Merton and Samuelson 1992.) Specifically, imagine that institutional arrangements are such that at each pre-retirement time s the agent is granted the right *and* obligation to nominate some future time $\overline{R}(s)$ whereupon he or she will permanently cease participating in the labor force. In other words, retirement precommitment amounts to a forward contract between employee and employer(s), rather than an option held by the employee.

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[1] On p.833 of the main text there were a few slips:

Equation (5) should read: $K(F,R) \equiv \max_{C,x} E_R\left[\int_R^T u(C(s),s)ds\right]$.

The lead-up to equation (7) should read:

$$\begin{aligned} J_R &= \frac{\partial}{\partial R}\left\{\max_{R,C,x} E_s\left[\int_s^R [u(C(t),t)-\ell(t)]dt + K(F,R)\right]\right\} \\ &= u(C^*,R^*) - \ell(R^*) + K_F(F,R^*)\cdot Y(R^*) - u(C^*,R^*) \\ &= 0 \end{aligned}$$

Finally, in the last sentence of footnote 4, "age-conditioned" should read "retirement-conditioned".

For simplicity, this appendix confines attention to the case of power consumption utility, i.e. $\frac{C^{g}-1}{g}$, g constant, and a constant, non-negative rate of time preference ρ, along with constancy of the wage rate Y and disutility of effort ℓ.

Consider the pre-retirement value function

$$L(F(s),s) = \max_{\overline{R},C,x} E_s \int_s^{\overline{R}} \left\{ \left[\left(\frac{C^{g}-1}{g} \right) - \ell \right] exp(-\,r\,t)dt + K\left(F\left(\overline{R}\right),\overline{R}\right) \right\} \qquad \text{(A1)}$$

where the constraint for $s < \overline{R}$ $\left[\equiv \overline{R}(s)\right]$ is

$$dF(s) = \left[\left(x(s)(a - r) + r \right) F(s) + Y - C(s) \right] ds + x(s)F(s)\,s\,dz(s)\,. \qquad \text{(A2)}$$

(In this way, a formal definition of retirement precommitment is now to hand). Recall that F, x, E_s, K, a, r, s and dz represent fungible assets, proportionate investment in risky assets, conditional expectations, post-retirement value function, expected return to risky assets, return to safe assets, volatility of returns to risky assets, and a Wiener increment. The first-order condition with respect to $\overline{R}$ is given by

$$-\ell e^{-r\overline{R}} + E_s \left[K_{F\left(\overline{R}\right)} \frac{\P F\left(\overline{R}\right)}{\P \overline{R}} \right] = 0\,, \qquad \text{(A3)}$$

that is,

$$-\ell + E_s \left\{ b\left(\overline{R}\right) \left[F\left(\overline{R}\right) \right]^{g-1} Y \right\} = 0\,. \qquad \text{(A4)}$$

In the case of an interior solution, equation (A4) yields the following closed form for the date of retirement under precommitment:

$$\overline{R} = T + n^{-1}\, ln \left\{ 1 - n \left(\frac{\ell}{Y} \right)^{\frac{1}{1-g}} \left[E_s \left[F\left(\overline{R}\right) \right]^{g-1} \right]^{\frac{1}{g-1}} \right\} \qquad \text{(A5)}$$

where the term $E_s \left[F\left(\overline{R}\right) \right]^{g-1}$ is evaluated below.

Remaining human capital at date *s*, $H\left(\overline{R},s\right)$, is given by

$$\left.\begin{aligned} H\left(\overline{R},s\right) &= \frac{Y}{r}\left[1-exp\left(r\left(s-\overline{R}\right)\right)\right] \text{for } 0 \le s < \overline{R} \\ &= 0 \qquad \text{for } \overline{R} \le s \le T. \end{aligned}\right\} \tag{A6}$$

This in conjunction with Merton (1969) gives an explicit value function — pre-retirement, and for the retirement timing problem under precommitment:

$$I\left(W,\overline{R},s\right) = \frac{b(s)}{g} exp\left(-\,rs\right)\left[W(s)\right]^{g} \;-\frac{\ell}{r}\left(exp\left(-\,rs\right)-exp\left(-\,r\overline{R}\right)\right) \tag{A7}$$

where the total-wealth state variable W is defined by

$$W = W\left(\overline{R},s\right) \equiv F(s) + H\left(\overline{R},s\right) \;, \tag{A8}$$

and $\overline{R}$ is yet to be fully evaluated.

Merton (1969) shows how to get from (A7) closed form solutions for optimal consumption $\overline{C}$, and for the optimal proportion of risky assets in total wealth, $\overline{x}$:

$$\overline{C} = \left[\frac{n}{1-exp\left(n\left(s-T\right)\right)}\right]W \;, \tag{A9}$$

$$\overline{x} = \frac{a\;-r}{s^{2}\left(1-g\right)} \qquad [= \text{constant}]. \tag{A10}$$

The human capital component of $W\left(\overline{R},s\right)$ declines through time until the retirement date is hit. Hence, equation (A10) justifies an *age-phased* solution to the pre-retirement asset allocation problem (Samuelson 1989). Indeed, the equation could justify heavy borrowing on the part of young households with a large and dependable earning capacity.

3. RETIREMENT TIMING UNDER PRECOMMITMENT

The next step is to evaluate $E_s\left[F\left(\bar{R}\right)\right]^{g-1}$. The basic idea comes from Merton (1971, Section 6). Begin with the transition equation for total wealth:

$$dW = \left[\left(\bar{x}(a - r) + r\right)W - \bar{C}\right]ds + \bar{x}\,s\,W dz\,. \tag{A11}$$

Application of equations (A9) and (A10) gives

$$\frac{dW}{W} = \left[\frac{(a-r)^2}{s^{\,2}(1-g)} + r - \frac{n}{1-exp(n\,(s-T))}\right]ds + \frac{(a-r)}{s\,(1-g)}dz\,. \tag{A12}$$

By Ito's Lemma, $dW/W = d\,ln\,W + \frac{1}{2}W^{-2}(dW)^2$, so (A12) can be written as

$$d\,lnW = \left\{\frac{(a-r)^2}{s^{\,2}(1-g)}\left[1-\frac{1}{2(1-g)}\right] + r - \frac{n}{1-exp(n\,(s-T))}\right\}ds + \frac{(a-r)}{s\,(1-g)}dz\,. \tag{A13}$$

Noting that the indefinite integral of $\frac{n}{1-exp(n\,(s-T))}$ is $ns - ln\{1-exp[n\,(s-T)]\}$, equation (A13) integrates up to

$$ln\,F\left(\bar{R}\right) = lnW + \left\{\frac{(a-r)^2}{s^{\,2}(1-g)}\left[1-\frac{1}{2(1-g)}\right] + r - n\right\}\left(\bar{R}-s\right)$$

$$+\,ln\left[\frac{1-exp\left(n\left(\bar{R}-T\right)\right)}{1-exp(n\,(s-T))}\right] + \frac{(a-r)}{s\,(1-g)}\int_s^{\bar{R}} dz\,\,. \tag{A14}$$

Next, multiply through by γ - 1, take exponentials, and run through the conditional expectations operator, to get

$$E_s\left[F\left(\bar{R}\right)\right]^{\gamma-1} = W^{\gamma-1}exp\left\{\left\{\frac{(\alpha-r)^2}{\sigma^2}\left[\frac{1}{2(1-\gamma)}-1\right] + (\gamma-1)(r-\nu)\right\}\left(\bar{R}-s\right)\right\}$$

$$\times\left[\frac{1-exp\left(n\left(\bar{R}-T\right)\right)}{1-exp(n\,(s-T))}\right]^{g-1}\cdot E_s\left\{exp\left[-\frac{(a-r)}{s}\int_s^{\bar{R}} dz\right]\right\}. \tag{A15}$$

Now properties of the standard-normal and log-normal distributions together imply

$$E_s\left\{exp\left[-\frac{(a-r)}{s}\int_s^{\bar{R}} dz\right]\right\} = exp\left[-\frac{(a-r)^2}{2s^2}\left(\bar{R}-s\right)\right]. \tag{A16}$$

This fact and the definition of ν enable us to express the harmonic mean $\left\{E_s\left[F(\bar{R})\right]^{g-1}\right\}^{\frac{1}{g-1}}$ as

$$\left\{E_s\left[F(\bar{R})\right]^{g-1}\right\}^{\frac{1}{g-1}} = W(s)exp\left[\left(\frac{r-r}{1-g}\right)\left(\bar{R}-s\right)\right]\left[\frac{1-exp\left(n\left(\bar{R}-T\right)\right)}{1-exp\left(n\left(s-T\right)\right)}\right] \tag{A17}$$

$$= n^{-1}\left(\frac{Y}{\ell}\right)^{\frac{1}{1-g}}\left\{1-exp\left[n\left(\bar{R}-T\right)\right]\right\} \tag{A18}$$

where equation (A18) uses (A5). Finally, cancel the term $\left[1-exp\left(n\left(\bar{R}-T\right)\right)\right]$ from both sides of (A18) and define $A(s) \equiv n^{-1}\left(\frac{Y}{\ell}\right)^{\frac{1}{1-g}}\left[1-exp\left(n\left(s-T\right)\right)\right]$ = reservation retirement assets, to obtain an implicit equation for $\bar{R}$ in terms of period-*s* magnitudes:

$$\left\{F(s)+\frac{Y}{r}\left[1-exp\left(r\left(s-\bar{R}\right)\right)\right]\right\}\cdot exp\left[\left(\frac{r-r}{1-g}\right)\left(s-\bar{R}\right)\right] = A(s) \tag{A19}$$

4. LOG CONSUMPTION UTILITY AND ZERO TIME PREFERENCE

A special case that was emphasized in the main text is defined by log consumption utility $(g \to 0)$ and zero time preference $(r = 0)$. In this case, equation (A19) simplifies to

$$\overline{R} = s + \frac{1}{r} ln \left(\frac{\frac{Y}{r} + A(s)}{\frac{Y}{r} + F(s)} \right) \tag{A20}$$

where $A(s) \equiv Y(T - s)/\ell$ = reservation retirement assets.

Note that the time to retirement, $\overline{R} - s$, is related to the extent that actual assets $F(s)$ fall short of reservation retirement assets, consistent with intuition. Note also that at the point of retirement $(s = R)$, (A20) reduces to

$$R = T - \frac{\ell F(R)}{Y} \tag{A21}$$

Equation (A21) is identical to the simple flexible-retirement formula that was emphasized in the main text.

5. DISCUSSION

The approximation procedure proposed in this appendix gives exact solutions at the time of retirement. Following Stock and Wise (1990) and Sundaresan and Zapatero (1997), the natural way to use this procedure for pre-retirement times is to assume that at each instant the agent behaves ‘as if’ he or she were solving the precommitment problem for the first and last time. In this way, triples $\left(\overline{C}(s), \overline{x}(s), \overline{R}(s)\right)$ can be calculated at each time $0 \le s \le R$.

How good is the approximation? By continuity, it will be excellent in the neighborhood of retirement. As we move back towards the start of working life, however, the approximation will progressively deteriorate, because an increasing proportion of the agent’s wealth consists of human capital. Missing from the approximations suggested here and elsewhere in the literature is the effect of expected dispersion in the date of retirement on the pre-retirement decisions of the agent. Each time your risky assets perform better (worse) than expected, you revise forward (backwards) your expected date of retirement. In this way, human capital is risky even if the agent's wage is deterministic, as was assumed herein. The dispersion effect will in general be heteroskedastic; as your expected date of retirement draws nearer, your remaining human capital tends to zero, resulting in less expected dispersion in your date of retirement, and greater accuracy of the foregoing approximate solution.

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